Integer Sequences Realized by the Subgroup Pattern of the Symmetric Group

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Abstract

The subgroup pattern of a finite group G is the table of marks of G together with a list of representatives of the conjugacy classes of subgroups of G. In this article we describe a collection of sequences realized by the subgroup pattern of the symmetric group.

1 Introduction

The table of marks of a finite group G was introduced by Burnside [1]. It is a matrix whose rows and columns are indexed by a list of representatives of the conjugacy classes of subgroups of G, where, for two subgroups $H, K \leq G$ the (H, K) entry in the table of marks of G is the number of fixed points of K in the transitive action of G on the cosets of H, $(\beta_{G/H}(K))$. If $H_1, \dots, H_r$ is a list of representatives of the conjugacy classes of subgroups of G, the table of marks is then the $(r \times r)$-matrix

$$M(G) = (\beta_{G/H_i}(H_j))_{i,j=1,\dots,r}.$$

In much the same fashion as the character table of G classifies matrix representations of G up to isomorphism, the table of marks of G classifies permutation representations of G up to equivalence. It also encodes a wealth of information about the subgroup lattice of G in a compact way. The `GAP` [3] library of tables of marks `Tomlib` [7] provides ready access to the tables of marks and conjugacy classes of subgroups of some 400 groups. The data exhibited in later sections has been computed using this library. The purpose of this article is to illustrate how interesting integer sequences related to the subgroup structure of a finite group can be computed from this data. This paper is organized as follows. In Section 2 we study the conjugacy classes of subgroups of S_n for $n \leq 13$. In Section 3 we examine the tables of marks of S_n for $n \leq 13$ and describe how much more information regarding the subgroup structure of S_n can be obtained. In Section 4 we discuss the Euler Transform and its applications in counting subgroups of S_n.

2 Counting Subgroups

Given a list of representatives $\{H_1, \dots, H_r\}$ of $\mathrm{Sub}(G)/G$, the conjugacy classes of subgroups of G, we can enumerate those subgroups which satisfy particular properties. The numbers of

conjugacy classes of subgroups of S_n and A_n are sequences A000638 and A029726 respectively in Sloane's encyclopedia [5].The `GAP` table of marks library `Tomlib` provides access to the conjugacy classes of subgroups of the symmetric and alternating groups for $n \le 13$. Table 1 records the number of conjugacy classes of subgroups of S_n which are abelian, cyclic, nilpotent, solvable and supersolvable, (SupSol). A similar table for the conjugacy classes of subgroups of the alternating groups can be found in Appendix A.

n	$\lvert\mathrm{Sub}(S_n)/S_n\rvert$	Abelian	Cyclic	Nilpotent	Solvable	SupSol
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	4	3	3	3	4	4
4	11	7	5	8	11	9
5	19	9	7	10	17	15
6	56	20	11	25	50	38
7	96	26	15	32	84	65
8	296	61	22	127	268	187
9	554	82	30	156	485	341
10	1593	180	42	531	1418	923
11	3094	236	56	648	2691	1789
12	10723	594	77	3727	9725	6118
13	20832	762	101	4221	18286	11616

Table 1: Sequences in S_n

2.1 Subgroup Orders

A question of historical interest concerns the orders of subgroups of S_n. In [2] Cameron writes: *The Grand Prix question of the Academie des Sciences, Paris, in 1860 asked "How many distinct values can a function of n variables take?" In other words what are the possible indices of subgroups of S_n.* For $n \le 13$ Table 2 records the numbers of different orders $\mathcal{O}(S_n), \mathcal{O}(A_n)$ of subgroups of S_n and A_n. One might as well also enumerate the number of "missing" subgroup orders, that is, the number, $d(S_n)$, of divisors d such that $d \mid |S_n|$ but S_n has no subgroup of order d. Table 3 records the number of missing subgroup orders of S_n and A_n for $n \le 13$.

n	$\mathcal{O}(S_n)$	$\mathcal{O}(A_n)$
1	1	1
2	2	1
3	4	2
4	8	5
5	13	9
6	21	15
7	31	22
8	49	38
9	74	59
10	113	89
11	139	115
12	216	180
13	268	226

Table 2: Subgroup Orders

n	$d(S_n)$	$d(A_n)$
1	0	0
2	0	0
3	0	0
4	0	1
5	3	3
6	9	9
7	29	26
8	47	46
9	86	81
10	157	151
11	401	365
12	576	540
13	1316	1214

Table 3: Missing Subgroup Orders

3 Counting Using the Table of Marks

If in addition to a list of conjugacy classes of subgroups of G, the table of marks of G is also available, or can be computed, one can say quite a lot about the structure of the lattice of subgroups of G. We begin this section by giving some basic information about tables of marks and then go on to describe how we can count incidences and edges in the lattice of subgroups.

3.1 About Tables of Marks

Let G be a finite group and let $\mathrm{Sub}(G)$ denote the set of subgroups of G. By $\mathrm{Sub}(G)/G$ we denote the set of conjugacy classes of subgroups of G. For $H, K \in \mathrm{Sub}(G)$ let

$$\beta_{G/H}(K) = \#\{Hg \in G/H : (Hg)k = Hg \text{ for all } k \in K\}$$

denote the mark of K on H. If $H_1, \ldots, H_r$ is a list of representatives of the conjugacy classes of subgroups of G, the table of marks is then the $(r \times r)$-matrix

$$M(G) = (\beta_{G/H_i}(H_j))_{i,j=1.,\ldots,r}.$$

The table of marks $M(S_4)$ of the symmetric group S_4 is shown in Figure 1.

	1	2	2	3	2^2	2^2	4	S_3	D_8	A_4	S_4
$S_4/1$	24										
$S_4/2$	12	4									
$S_4/2$	12	.	2								
$S_4/3$	8	.	.	2							
$S_4/2^2$	6	6	.	.	6						
$S_4/2^2$	6	2	2	.	.	6					
$S_4/4$	6	2	.	.	.	.	2				
S_4/S_3	4	.	2	1	.	.	.	1			
S_4/D_8	3	3	1	.	3	1	1	.	1		
S_4/A_4	2	2	.	2	2	.	.	.	.	2	
S_4/S_4	1	1	1	1	1	1	1	1	1	1	1

Figure 1: Table of Marks $M(S_4)$

As a matrix, we can extract a variety of sequences from the table of marks, the most obvious of which is the sum of the entries. The sum of the entries of $M(S_n)$ for $n \leq 13$ is shown in Figure 4. We can also sum the entries on the diagonal to obtain the sequences in Figure 5.

n	S_n	A_n
1	1	1
2	4	1
3	18	5
4	146	39
5	681	192
6	7518	1717
7	58633	13946
8	952826	243391
9	11168496	2693043
10	232255571	38343715
11	3476965896	545787051
12	108673489373	15787210045

Table 4: Sum of $M(G)$

n	S_n	A_n
1	1	1
2	3	1
3	10	4
4	47	19
5	165	73
6	950	412
7	5632	2660
8	43772	21449
9	376586	184541
10	3717663	1827841
11	40555909	20043736
12	484838080	240206213

Table 5: Sum of the Diagonal

We will now collect some elementary properties of tables of marks in Lemma 1.

Lemma 1. *Let $H, K \leq G$. Then the following hold:*

(i) The first entry of every row of $M(G)$ is the index of the corresponding subgroup,

$$\beta_{G/K}(1) = |G : K|.$$

(ii) The entry on the diagonal is,

$$\beta_{G/K}(K) = |N_G(K) : K|.$$

(iii) The length of the conjugacy class $[K]$ *of* K *is given by,*

$$|[K]| = |G : N_G(K)| = \frac{\beta_{G/K}(1)}{\beta_{G/K}(K)}.$$

(iv) The number of conjugates of K *which contain* H *is given by,*

$$|\{K^a | a \in G, H \leq K^a\}| = \frac{\beta_{G/K}(H)}{\beta_{G/K}(K)}.$$

The following formula which follows trivially from Lemma 1 (iv) relates marks to incidences in the subgroup lattice of G.

$$\beta_{G/K}(H) = |N_G(K) : K| \cdot \#\{K^g : H \leq K^g, g \in G\}. \tag{1}$$

As a first application of Formula 1 we obtain the following lemma which enables us to count the total number of subgroups of G.

Lemma 2. *Given a list* $\{H_1, \ldots, H_r\}$ *of representatives of the conjugacy classes of subgroups of* G*, the total number of subgroups of* G *is*

$$|\mathrm{Sub}(G)| = \sum_{i=1}^{r} \frac{\beta_{G/H_i}(1)}{\beta_{G/H_i}(H_i)}.$$

Proof. It follows from Formula 1 that for any subgroup $H \leq G$, $\frac{\beta_{G/H}(1)}{\beta_{G/H}(H)}$ is the length of the conjugacy class of H in G. $\square$

Table 6 lists the total number of subgroups of S_n and A_n for $n \leq 13$, sequences A005432 and A029725.

n	A_n	S_n
1	1	1
2	1	2
3	2	6
4	10	30
5	59	156
6	501	1455
7	3786	11300
8	48337	151221
9	508402	1694723
10	6469142	29594446
11	81711572	404126228
12	2019160542	10594925360
13	31945830446	175238308453

Table 6: Total Number of Subgroups of A_n and S_n

3.2 Counting Incidences

Another immediate consequence of Formula 1 is that by dividing each row of the table of marks of G by its diagonal entry $\beta_{G/H_i}(H_j)$ we obtain a matrix $\mathcal{C}(G)$ describing containments in the subgroup lattice of G, where the (H, K)-entry is

$$\mathcal{C}(H, K) = \#\{K^g : H \leq K^g, g \in G\}. \tag{2}$$

Figure 2 illustrates the containment matrix of the symmetric group S_4.

$$\left(\begin{array}{c|ccccccccccc}
1 & 1 \\
2 & 3 & 1 \\
2 & 6 & . & 1 \\
3 & 4 & . & . & 1 \\
2^2 & 1 & 1 & . & . & 1 \\
2^2 & 3 & 1 & 1 & . & . & 1 \\
4 & 3 & 1 & . & . & . & . & 1 \\
S_3 & 4 & . & 2 & 1 & . & . & . & 1 \\
D_8 & 3 & 3 & 1 & . & 3 & 1 & 1 & . & 1 \\
A_4 & 1 & 1 & . & 1 & 1 & . & . & . & . & 1 \\
S_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 2 & 2 & 3 & 2^2 & 2^2 & 4 & S_3 & D_8 & A_4 & S_4
\end{array}\right)$$

Figure 2: Containment Matrix : $\mathcal{C}(S_4)$

Given the containment matrix $\mathcal{C}(G)$ we can easily obtain the incidence matrix, $\mathcal{I}(G)$, of the poset of conjugacy classes of subgroups of G by replacing each nonzero entry in $\mathcal{C}(G)$, (or indeed in $M(G)$) by an entry 1, where $\mathcal{I}(H, K) = 1$ if and only if K is subconjugate to H in G. Figure 3 shows the incidence matrix $\mathcal{I}(S_4)$ of the poset of conjugacy classes of subgroups of S_4.

$$
\left(\begin{array}{c|ccccccccccc}
1 & 1 & & & & & & & & & & \\
2 & 1 & 1 & & & & & & & & & \\
2 & 1 & . & 1 & & & & & & & & \\
3 & 1 & . & . & 1 & & & & & & & \\
2^2 & 1 & 1 & . & . & 1 & & & & & & \\
2^2 & 1 & 1 & 1 & . & . & 1 & & & & & \\
4 & 1 & 1 & . & . & . & . & 1 & & & & \\
S_3 & 1 & . & 1 & 1 & . & . & . & 1 & & & \\
D_8 & 1 & 1 & 1 & . & 1 & 1 & 1 & . & 1 & & \\
A_4 & 1 & 1 & . & 1 & 1 & . & . & . & . & 1 & \\
S_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 2 & 2 & 3 & 2^2 & 2^2 & 4 & S_3 & D_8 & A_4 & S_4
\end{array}\right)
$$

Figure 3: Incidence Matrix : $\mathcal{I}(S_4)$

For comparison with Figure 3 we illustrate the poset of conjugacy classes of subgroups of S_4 in Figure 4.

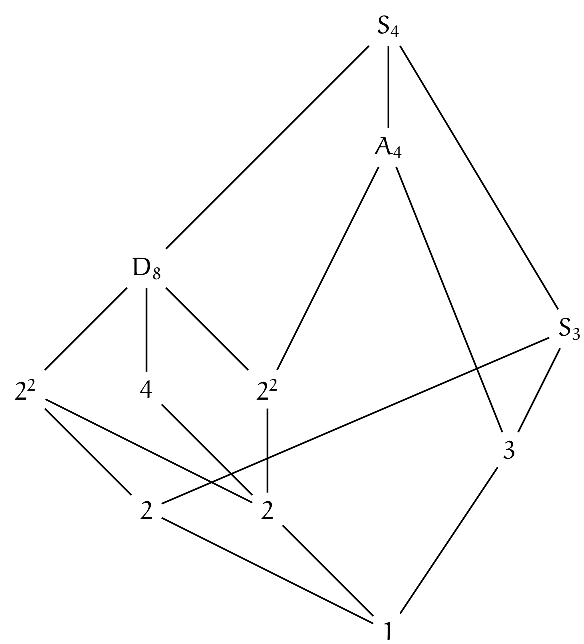

Figure 4: Poset of Conjugacy Classes of Subgroups of S_4

Lemma 3. *The number of incidences in the poset of conjugacy classes of subgroups of* G *is given by*

$$\sum \mathcal{I}(G).$$

Proof. The incidence matrix $\mathcal{I}(G)$ is obtained by replacing every nonzero entry in the table of marks by an entry 1. By Formula 1 $\mathcal{I}(H, K) = 1$ if and only if K is subconjugate to H in G, i.e. if and only if H and K are incident in the poset of conjugacy classes of subgroups of G. □

Figure 7 lists the number of incidences in the poset of conjugacy classes of subgroups of A_n and S_n for $n \leq 13$.

Lemma 4. *The total number of incidences in the entire subgroup lattice of* G *is given by*

$$\sum \mathcal{C}(G).$$

Proof. For $H, K \in \mathrm{Sub}(G)/G$ the H, K entry in $\mathcal{C}(G)$ is the number of incidences between H, K in the subgroup lattice of G. Thus summing over the entries in $\mathcal{C}(G)$ yields the total number of incidences in the entire subgroup lattice of G. □

Table 8 records the number of incidences in the subgroup lattices of S_n and A_n for $n \leq 13$.

n	$\sum \mathcal{I}(S_n)$	$\sum \mathcal{I}(A_n)$
1	1	1
2	3	1
3	9	3
4	44	13
5	101	32
6	523	128
7	1195	330
8	6751	2309
9	16986	4271
10	87884	12468
11	248635	33329
12	1709781	196182
13	4665651	490137

Table 7: Incidences in Poset

n	A_n	S_n
1	1	1
2	1	3
3	3	11
4	18	68
5	85	262
6	657	2261
7	4374	14032
8	55711	176245
9	530502	1821103
10	6603007	30883491
11	82736601	415843982
12	2032940127	10779423937
13	32102236563	177718085432

Table 8: Incidences in Subgroup Lattice

3.3 Counting Edges in Hasse Diagrams

The table of marks also allows us to count the number of edges in both the Hasse diagrams of the poset of conjugacy classes of subgroups and the subgroup lattice of G. To compute such data requires careful analysis of maximal subgroups in the subgroup lattice.

Formula 1 describes containments in the poset of conjugacy classes of subgroups looking upward through the subgroup lattice of G. But we can also view marks as containments looking downward through the subgroup lattice of G.

Lemma 5. *Let* $H, K \in \mathrm{Sub}(G)/G$. *Then the number of conjugates of* H *contained in* K *is given by*

$$E^{\uparrow}(H,K) = |\{H^g, g \in G : H^g \leq K\}| = \frac{\beta_{G/K}(H)\beta_{G/H}(1)}{\beta_{G/H}(H)\beta_{G/K}(1)}$$

Proof. The total number of edges between the classes $[H]_G$ and $[K]_G$ can be counted in two different ways, as the length of the class times the number of edges leaving one member of the class. Thus

$$|[H_G| \cdot |\{H^g, g \in G : H^g \leq K\}| = |[K]_G| \cdot |\{K^g, g \in G : K^g \geq H\}|.$$

By Formula 1 $|[H]_G| = \frac{\beta_{G/H}(1)}{\beta_{G/H}(H)}$ and $|[K]_G| = \frac{\beta_{G/K}(1)}{\beta_{G/K}(K)}$. Thus $E^{\uparrow}(H,K)$ can be expressed in terms of marks by Formula 1. □

3.3.1 Identifying Maximal Subgroups

It will be necessary, for the sections that follow, to identify for $H_i \in \mathrm{Sub}(G)/G$ which classes $H_j \in \mathrm{Sub}(G)/G$ are maximal in H_i.

Lemma 6. *Let* $H_i \in \mathrm{Sub}(G)/G = H_1, \ldots, H_r$. *Denote by* $\rho_i = \{j : H_j <_G H_i\}$ *the set of indices in* $\{1, \ldots, r\}$ *of proper subgroups of* H_i *up to conjugacy in* G. *Then the positions of all maximal subgroups of* H_i *are given by*

$$\mathrm{Max}(H_i) = \rho_i \setminus \bigcup_{j \in \rho_i} \rho_j \tag{3}$$

The set of values ρ_i are easily read off the table of marks of G by simply identifying the nonzero entries in the row corresponding to G/H_i. Formula 3 is implemented in `GAP` via the function `MaximalSubgroupsTom`.

Lemma 7. *Let* $\mathrm{Sub}(G)/G = H_1, \ldots, H_r$ *be a list of representatives of the conjugacy classes of subgroups of* G. *The number of edges in the Hasse diagram of the poset of conjugacy classes of subgroups of* G *is given by*

$$|E(\mathrm{Sub}(G)/G)| = \sum_{i=1}^{r} |\mathrm{Max}(H_i)|.$$

Proof. By Lemma 6 $\mathrm{Max}(H_i)$ is a list of the positions of the maximal subgroups of H_i up to conjugacy in G. In the Hasse diagram of the poset $\mathrm{Sub}(G)/G$ each edge corresponds to a maximal subgroup. □

Table 9 records the number of edges in the hasse diagram of the poset of conjugacy classes of subgroups of S_n and A_n for $n \leq 13$. In order to count the number of edges in the hasse diagram of the entire subgroup lattice of G we appeal to Formula 1 and Lemma 5.

Lemma 8. *Let* $\mathrm{Sub}(G)/G = H_1, \ldots, H_r$ *be as above. The total number of edges* $E(L(G))$ *in the Hasse diagram of the subgroup lattice of* G *is given by*

$$E(L(G)) = \sum_{i=1}^{r} \sum_{j \in \mathrm{Max}(H_i)} E^{\uparrow}(H_i, H_j).$$

Proof. By restricting $E^{\uparrow}(H_i, H_j)$ to those classes H_i, H_j which are maximal we obtain the number of edges connecting maximal subgroups of G. $\square$

Table 10 records the total number of edges in the hasse diagram of the subgroup lattice of S_n and A_n for $n \leq 13$.

n	$\lvert E(S_n)\rvert$	$\lvert E(A_n)\rvert$
1	0	0
2	1	0
3	4	1
4	17	5
5	37	13
6	149	44
7	290	98
8	1080	419
9	2267	722
10	8023	1592
11	17249	3304
12	72390	12645
13	153419	24792

Table 9: Edges in Poset

n	A_n	S_n
1	0	0
2	0	1
3	1	8
4	15	66
5	168	501
6	2051	6469
7	19305	60428
8	283258	926743
9	3255913	11902600
10	46464854	240066343
11	670282962	3677270225
12	18723796793	108748156239
13	321480817412	1980478458627

Table 10: Edges in subgroup lattice

3.4 Maximal Property-P Subgroups

For any property P which is inherited by subgroups of G we can use the table of marks of G to enumerate the maximal property P subgroups of G.

Lemma 9. *Let* $\mathrm{Sub}(G)/G = H_1, \ldots, H_r$ *and let* $\rho = \{i \in [1, \ldots, r] : H_i$ *is a property* P *subgroup*$\}$. *Then the positions of the maximal property* P *subgroups of* G *are given by*

$$P(G) = \rho \setminus \bigcup_{j \in \rho} \mathrm{Max}(H_j) \tag{4}$$

Figure 5 illustrates the maximal abelian subgroups of S_4.

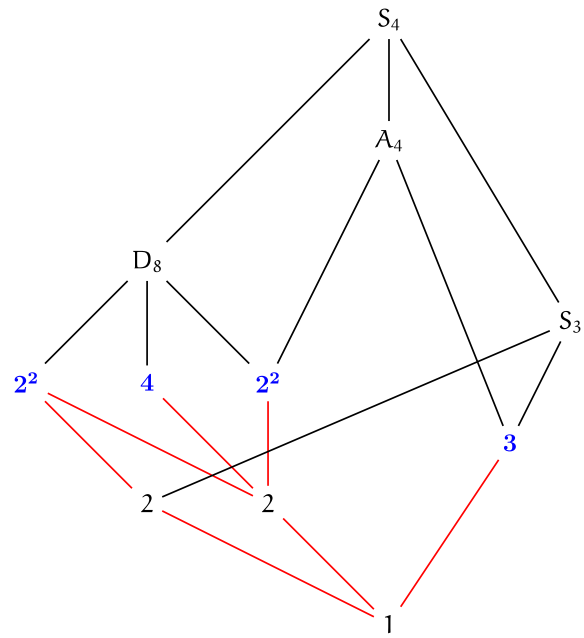

Figure 5: Maximal Abelian Subgroups of S_4

Table 11 records, for each of the properties listed across the first row of the table, the numbers of maximal property P classes of subgroups of S_n. A similar table for the alternating groups can be found in the Appendix.

n	Solvable	SupSol	Abelian	Cyclic	Nilpotent
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	2	2	2
4	1	2	4	3	2
5	3	3	5	3	3
6	4	4	7	5	5
7	5	5	10	6	6
8	6	6	17	11	7
9	9	8	23	15	9
10	12	11	30	20	12
11	14	14	41	24	15
12	17	19	61	34	20
13	24	23	80	43	25

Table 11: Maximal Property-P Subgroups of S_n

4 Connected Subgroups and the Euler Transform

The conjugacy classes of subgroups of the symmetric group play an important role in the theory of combinatorial species as described in [4]. Permutation groups have been used to answer many questions about species. Every species is the sum of its molecular subspecies. These molecular species correspond to conjugacy classes of subgroups of $\mathsf{Sym}(n)$. Molecular species decompose as products of atomic species which in turn correspond to connected subgroups of $\mathsf{Sym}(n)$ in the following sense.

Definition 1. *For each* $H \le \mathsf{Sym}(X)$ *there is a finest partition of* $X = \sqcup Y_i$ *such that* $H \le \prod \mathsf{Sym}(Y_i)$*, i.e.* $H = \prod H_i$ *with* $H_i \le \mathsf{Sym}(Y_i)$*. We allow* $H_i = 1$ *when* $|Y_i| = 1$*. We say that* H *is a connected subgroup of* $\mathsf{Sym}(X)$ *if the finest partition is* X*.*

In general a subgroup $H \le \mathsf{Sym}(X) = \prod H_i$ is a product of connected subgroups H_i. Sequence A000638 records the number of molecular species of degree n or equivalently the number of conjugacy classes of subgroups of $\mathsf{Sym}(n)$. Sequence A005226 records the number of atomic species of degree n or equivalently the number of connected conjugacy classes of subgroups of $\mathsf{Sym}(n)$.

Lemma 10. *There is a bijection between the conjugacy classes of subgroups of* S_n *and the set of pairs of the form* $(\lambda, (C_1, \dots, C_n))$ *where* $\lambda = 1^{a_1}, 2^{a_2}, \dots, n^{a_n}$ *is a partition of* n *and* C_i *is a multiset of* a_i *conjugacy classes of connected subgroups of* S_i *for* $i = 1, \dots, n$*.*

Proof. Given a representative H of the conjugacy class of subgroups $[H] \in \mathsf{Sub}(S_n)/S_n$ we associate a pair $(\lambda, (C_1, \dots, C_n))$ to H as follows. Write $H = \prod H_k$ where H_k is a connected subgroup of $\mathsf{Sym}(Y_k)$. Then $X = \{1, \dots, n\} = \sqcup Y_k$. Recording the size of each Y_k yields a partition $\lambda = 1^{a_1}, 2^{a_2}, \dots, n^{a_n}$. For $1 \le i \le n, C_i$ is a multiset of S_i-classes of subgroups H_k with $|Y_k| = i$. Bijectivity follows from the fact that conjugate subgroups yield the same λ and since $H^g = \prod H_k^g$, conjugate subgroups yield conjugate C_i. □

4.1 The Euler Transform

If two sequences $\{m_n\} = m_1, m_2, m_3, \dots$ and $\{c_k\} = c_1.c_2, c_3, \dots$ are related by

$$1 + \sum_{n \ge 1} m_n x^n = \prod_{k \ge 1} \left(\frac{1}{1 - x^k} \right)^{c_k}, \tag{5}$$

then we say that $\{m_n\}$ is the Euler transform of $\{c_k\}$ and that $\{c_k\}$ is the inverse Euler transform of $\{m_n\}$. There are many applications of this pair of transforms (see [6]). For example, the inverse Euler transform applied to the sequence of unlabeled graphs on n nodes (A000088) yields the sequence of connected graphs on n nodes (A001349). The inverse Euler transform of A000638 (the number of conjugacy classes of subgroups of S_n) is A005226, (the number of connected conjugacy classes of subgroups of S_n). To see why this is so we appeal to Definition 1, Lemma 10 and note that the coefficient of x^n in the product on the left hand side of Formula 5 is

$$m_n = \sum_{1^{a_1}, 2^{a_2}, \dots, n^{a_n} \vdash n} \prod_i \left(\!\!\binom{c_i}{a_i}\!\!\right) \tag{6}$$

i.e. the number of a_i-element multisets chosen from a set of c_i objects.

4.2 Counting subgroups of the Alternating Group

In Section 4 we noted that molecular species correspond to conjugacy classes of subgroups of $\mathsf{Sym}(n)$ and that atomic species correspond to conjugacy classes of connected subgroups of $\mathsf{Sym}(n)$ in the sense of Definition 1. In this Section we will count the number of conjugacy classes of subgroups of A_n which correspond to molecular and atomic species, and also count the number of conjugacy classes of connected subgroups of A_n.

4.2.1 Species based on Subgroups of the Alternating Group

In order to count molecular and atomic species corresponding to subgroups of A_n it will be necessary to introduce the following terminology. For a finite group G let $\mathsf{Sub}(G)$ denote the set of subgroups of G and let $\mathsf{Sub}(G)/G$ denote the conjugacy classes of subgroups of G. For the symmetric group we distinguish between two types of subgroups of S_n. The subgroups of A_n will be called *blue subgroups* and the subgroups of S_n which are not contained in A_n will be called *red subgroups*. The set of subgroups of S_n then is the disjoint union

$$\mathsf{Sub}(S_n) = \mathcal{B} \sqcup \mathcal{R}$$

where

$$\mathcal{B} = \mathsf{Sub}(A_n),\ \mathcal{R} = \mathsf{Sub}(S_n) \setminus \mathsf{Sub}(A_n)$$

Since no red subgroup is conjugate to a blue subgroup, both $\mathcal{B}$ and $\mathcal{R}$ are S_n-sets. The conjugacy classes of subgroups of S_n are then

$$\mathsf{Sub}(S_n)/S_n = \mathcal{B}/S_n \sqcup \mathcal{R}/S_n$$

Since molecular species correspond to S_n-sets we see that $\mathcal{B}/S_n$ is the set of conjugacy classes of subgroups of A_n which correspond to molecular species and that $\mathcal{R}/S_n$ is the set of conjugacy classes of subgroups of S_n, not contained in A_n, which correspond to molecular species. Table 12 illustrates both of these sequences together with the numbers of conjugacy classes of subgroups of S_n and A_n. For $n \in 1, \ldots, 13$ we see that $|\mathsf{Sub}(S_n)/S_n| = |\mathcal{B}/S_n| + |\mathcal{R}/S_n|$.

Since species correspond to S_n-orbits. In order to count the number of atomic species corresponding to conjugacy clases of subgroups of A_n we restrict our attention to the set $\mathcal{B}/S_n$. We can also count the number of atomic species corresponding to subgroups of S_n not contained in A_n by analysing $\mathcal{R}/S_n$. It should now be clear that in order to count the number of atomic species corresponding to conjugacy classes of subgroups of A_n we apply the Inverse Euler transform to the sequence $|\mathcal{B}/S_n|$ in Table 12 to obtain

$$1, 0, 1, 3, 4, 12, 12, 65, 58, 167, 198, 1207, 1178.$$

And to count the number of atomic species corresponding to conjugacy classes of red subgroups of S_n we examine $\mathcal{R}/S_n$ in GAP to obtain

$$0, 1, 1, 3, 2, 15, 8, 65, 66, 431, 443, 3643, 3594$$

n	$\lvert\mathsf{Sub}(S_n)/S_n\rvert$	$\lvert\mathsf{Sub}(A_n)/A_n\rvert$	$\lvert\mathcal{B}/S_n\rvert$	$\lvert\mathcal{R}/S_n\rvert$
1	1	1	1	0
2	2	1	1	1
3	4	2	2	2
4	11	5	5	6
5	19	9	9	10
6	56	22	22	34
7	96	40	37	59
8	296	137	112	184
9	554	223	195	359
10	1593	430	423	1170
11	3094	788	780	2314
12	10723	2537	2401	8322
13	20832	4558	4409	16423

Table 12: Red and Blue Subgroups of S_n

4.2.2 Connected Subgroups of the Alternatng Group

When we consider counting the connected subgroups of A_n we turn our attention to the list of conjugacy classes of subgroups of A_n (A029726). By analyzing each of the subgroups in turn using GAP we obtain the sequences

$$1, 0, 1, 3, 4, 12, 15, 87, 64, 168, 205, 1336, 1198$$

Remark 2. *There is a sequence in the encyclopedia, A116653, which claims to count both the number of atomic species based on subgroups of the alternating group and the number of connected subgroups of A_n. This sequence is simply the inverse Euler transform of sequence A029726, the number of conjugacy clases of subgroups of the alternating group. It should be clear that our sequences actually count the number of objects claimed.*

4.3 Connected Subgroups with Additional Properties

Appealing to Definition 1 we can count the connected subgroups of S_n which possess additional group theoretic properties. We must be mindful of the fact that these additional properties should be compatible with taking direct products. Table 13 records the number of connected subgroups of S_n which additionally possess the properties listed in the first row of the table.

n	$\lvert Sub(S_n)/S_n \rvert$	Abelian	Nilpotent	Solvable	SupSol
1	1	1	1	1	1
2	2	1	1	1	1
3	4	1	1	2	2
4	11	3	4	6	4
5	19	1	1	4	4
6	56	6	9	23	15
7	96	1	1	16	13
8	296	17	69	122	81
9	554	5	8	109	77
10	1593	40	238	551	352
11	3094	2	2	570	406
12	10723	162	2339	4633	2995
13	20832	5	8	4224	2866

Table 13: Connected Subgroups of S_n

Each of the sequences in Table 13 is the inverse Euler transform of the corresponding sequence in Table 1.

4.4 Connected Partitions

Since the direct product of two cyclic groups is not, in general, a cyclic group there is no column in Table 13 corresponding to cyclic groups. The inverse Euler transform applied to the sequence of conjugacy classes of cyclic subgroups of S_n yields the all ones sequence. We can, however, use GAP to count the number of connected cyclic subgroups of S_n to obtain

$$1, 1, 1, 2, 1, 4, 1, 5, 3, 8, 2, 14, 3$$

This sequence is quite close to two sequences already contained in the encyclopedia, sequences A018783 and A200976. What we are in fact counting is the number of connected partitions of n.

As an example consider the symmetric group S_{13} which has 101 conjugacy classes of cyclic subgroups. Of those 101 conjugacy classes of subgroups, only 3 are not decomposable as products. Representatives of their generators are shown below.

C_n	Generator	λ
6	$(1,4)(2,3)(5,6,7)(8,13,10,12,9,11)$	$[2,2,3,6]$
12	$(1,4,5,2,3,6)(7,8,9)(10,12,11,13)$	$[6,3,4]$
13	$(1,2,3,4,5,6,7,8,9,10,11,12,13)$	$[13]$

Table 14: Partitions of 13

The partitions described above can be visualized as graphs where the vertices are represented by the lengths of the cycles and two cycles are connected if and only if there gcd is > 1. Figure 6 illustrates the three connected partitions of 13 from Table 14.

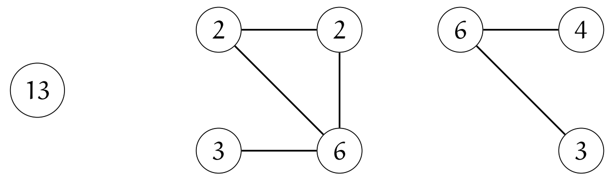

Figure 6: Partitions of 13

Sequence A018783 counts the number of partitions of n into parts having a common factor, while sequence A200976 counts the number of partitions of n such that each pair of parts (if any) has a common factor.

Appendix A Sequences

For any of the sequences above which count conjugacy clases of subgroups we can use the table of marks of S_n or A_n to count the total number of subgroups.

B Sequences in the symmetric group

n	$\lvert \mathrm{Sub}(S_n) \rvert$	Abelian	Cyclic	Nilpotent	Solvable	SupSol
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	6	5	5	5	6	6
4	30	21	17	24	30	28
5	156	87	67	102	154	144
6	1455	612	362	837	1429	1259
7	11300	3649	2039	5119	11065	9560
8	151221	35515	14170	78670	148817	123102
9	1694723	289927	109694	664658	1667697	1371022
10	29594446	3771118	976412	13514453	29103894	23449585
11	404126228	36947363	8921002	137227213	396571224	317178020
12	10594925360	657510251	101134244	4919721831	10450152905	8296640115
13	175238308453	7736272845	1104940280	60598902665	172658168937	136245390535

Table 15: Total no of subgroups of S_n

n	Solvable	SupSol	Abelian	Cyclic	Nilpotent
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	4	4	4
4	1	7	11	13	7
5	21	31	51	31	31
6	76	101	241	246	211
7	456	491	1506	1296	1156
8	1956	3011	9649	10774	5419
9	12136	18467	80281	83238	40027
10	80836	114983	640741	788820	348331
11	807676	1283723	6196576	6835170	3204796
12	8779816	13380643	66883411	81364944	38422891
13	104127596	148321603	775421219	848378532	467645179

Table 16: Total Number of Maximal Property-P Subgroups of S_n

C Sequences in the alternating group

n	$\lvert \mathrm{Sub}(A_n)/A_n \rvert$	Abelian	Cyclic	Nilpotent	Solvable	SupSol
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	2	2	2	2	2	2
4	5	4	3	4	5	4
5	9	5	4	5	8	7
6	22	9	6	10	19	14
7	40	12	8	13	33	22
8	137	30	12	53	122	70
9	223	41	17	69	192	122
10	430	60	23	122	364	225
11	788	81	29	160	650	395
12	2537	193	40	734	2194	1240
13	4558	243	52	848	3845	2185

Table 17: Conjugacy classes of subgroups of A_n

n	$\lvert\mathrm{Sub}(A_n)\rvert$	Abelian	Cyclic	Nilpotent	Solvable	SupSol
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	2	2	2	2	2	2
4	10	9	8	9	10	9
5	59	37	32	37	58	53
6	501	207	167	252	488	418
7	3786	1192	947	1507	3664	2894
8	48337	11449	6974	21739	47210	33675
9	508402	93673	53426	186983	498102	369763
10	6469142	892783	454682	2369258	6293475	4769542
11	81711572	8534308	4303532	22872863	78805290	58853842
12	2019160542	148561283	50366912	746597568	1960342409	1395051100
13	31945830446	1740198891	553031624	9157758326	31130243721	21847262156

Table 18: Total no of subgroups of A_n

n	Solvable	SupSol	Abelian	Cyclic	Nilpotent
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	2	2	2	2
5	3	3	3	3	3
6	4	3	5	4	3
7	5	4	6	5	5
8	6	6	13	6	6
9	10	8	19	8	7
10	12	10	22	10	9
11	14	13	27	14	12
12	17	18	40	20	17
13	24	22	54	24	20

Table 19: Maximal Property-P Subgroups of A_n

n	$\lvert Sub(A_n)/A_n \rvert$	Abelian	Nilpotent	Solvable	SupSol
1	1	1	1	1	1
2	1	0	0	0	0
3	2	1	1	1	1
4	5	2	2	3	2
5	9	1	1	3	3
6	22	3	4	10	6
7	40	1	1	11	6
8	137	14	36	80	42
9	223	5	9	52	39
10	430	12	49	145	85
11	788	2	2	165	104
12	2537	69	489	1208	686
13	4558	3	4	1033	617

Table 20: Connected Subgroups of A_n

The Connected even partitions of n

$$1, 0, 1, 1, 1, 2, 1, 3, 3, 4, 2, 8, 2$$

n	Solvable	SupSol	Abelian	Cyclic	Nilpotent
1	1	1	1	1	1
2	1	1	1	1	1
3	3	3	3	3	3
4	1	10	10	9	10
5	36	40	30	30	30
6	225	110	115	100	110
7	686	645	861	665	1001
8	4655	5670	10536	3885	4005
9	28728	47754	78474	33093	45696
10	397005	311850	1008000	371700	379155
11	2210890	3014550	9302964	3790875	4913040
12	26975025	24022845	73024380	37839285	36701280
13	26121667	46950904	563291872	350984414	158538380

Table 21: Total Number of Maximal Property-P Subgroups of A_n

References

[1] W. Burnside, *Theory of groups of finite order*, Dover Publications Inc., New York, 1955, 2d ed. MR 0069818 (16,1086c)

[2] Peter J. Cameron, *Permutation groups*, London Mathematical Society Student Texts, vol. 45, Cambridge University Press, Cambridge, 1999. MR 1721031 (2001c:20008)

[3] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008.

[4] J. Labelle and Y. N. Yeh, *The relation between Burnside rings and combinatorial species*, J. Combin. Theory Ser. A **50** (1989), no. 2, 269–284. MR 989198 (90c:05016)

[5] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, Notices Amer. Math. Soc. **50** (2003), no. 8, 912–915. MR 1992789 (2004f:11151)

[6] N. J. A. Sloane and Simon Plouffe, *The encyclopedia of integer sequences*, Academic Press Inc., San Diego, CA, 1995, With a separately available computer disk. MR 1327059 (96a:11001)

[7] *Tomlib, Version 1.2.1* , 2011, `http://schmidt.nuigalway.ie/tomlib`.